

On a relation between Liouville field theory and a two component scalar field theory passing through the random walk

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In this work it is proposed a transformation which is useful in order to simplify non-polynomial potentials given in the form of an exponential. As an application, it is shown that the quantum Liouville field theory may be mapped into a field theory with a polynomial interaction between two scalar fields and a massive vector field.

I. INTRODUCTION

The Liouville field theory is actively studied both in physics and mathematics [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. Several approaches have been proposed for its quantization, see for example Refs. [3, 15] for a list of references on this subject.

Motivated by the difficulties encountered in carrying out the quantization program of the Liouville model posed by the exponential interaction term present in its action, we show here the equivalence between Liouville field theory and a field theory with polynomial action which describes the interaction of two scalar fields with a massive vector field. The original degrees of freedom of the Liouville model are conveniently mapped into the longitudinal component of the vector field, while the spurious transverse components decouple after taking a suitable limit of large coupling constant. The appearance of the mass term of the vector field is related to the kinetic energy of the Liouville field theory. The two scalar fields are instead associated to constraints, so that they do not introduce other, unwanted, degrees of freedom. The advantage of the new theory derived in this way is that it has a structure similar to

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that of massive scalar electrodynamics and it is thus much more tractable than Liouville field theory with standard techniques, like for instance the perturbative approach [35].

At the heart of the mapping of the Liouville model into a polynomial field theory there is a transformation which allows to rewrite the exponential term typical of Liouville field theory in the form of the generating functional of a two component scalar field theory. The price to be paid is the introduction of new degrees of freedom, however the scalar sector of the theory is almost trivial because, as already mentioned, its role is just to impose constraints. Solvable theories in which the number of degrees of freedom is zero have been known for more than thirty years [19]. The technique of expressing complicated potentials in terms of amplitudes of almost trivial field theories has been presented in Ref. [16] in the case of topological interactions in polymer physics. The advantages of this technique in modeling the entanglement of two or more polymers have been explained in Refs. [17, 18].

The mapping of Liouville field theory into the two component scalar field theory coupled to a vector field is based in part on known results of the theory of Brownian motion. In particular, it is applied the field theoretical representation of the grand canonical partition function of charged particles subjected to a random walk while immersed in a magnetic field. This is not the first time that Liouville field theory has been related to statistical systems. For example, in the works of Refs. [5, 6] the theory of Liouville appears either like a random field or as a mean to express the normalization of the ground state of a model of disordered conductors. In [7, 21] it is associated to quantum chaos in anyon systems. However, the main interest of this work is not the connection between Liouville field theory and statistical mechanics. The goal is instead to map the Liouville field theory into a theory which is simpler in the sense that its action is a polynomial in the fields, so that field theoretical techniques like the perturbative approach can be applied to it. The statistical theory of the random walk enters in our approach because it offers a nice way to express the Green function of fields coupled to vector fields, a fact already noted in Ref. [24]. The application of these ideas to Liouville field theory and the derivation of the equivalent model with polynomial interactions constitutes the original part of the present work.

This paper is organized as follows. In Section II are presented the basic facts of the statistical mechanics of Brownian motion which will be used later. In Section III the partition function of a two component scalar field theory interacting with a massive vector field is introduced using a path integral formulation. In the limit in which the strength of the

interactions of the massive vector fields becomes infinite, their transverse degrees of freedom decouple from the other fields and only the longitudinal component survives. In Section IV the two scalar fields are integrated out from the partition function. The result is a field theory in which there is an unique scalar field with an exponential interaction term. The equivalence of this field theory with the model of Liouville is proved in Section V. Our conclusions are drawn in Section VI. Finally, in the Appendix a procedure to integrate out the scalar fields is presented, which is alternative to that of Section IV. It is shown that the results of both procedures coincide.

II. THE GRAND CANONICAL PARTITION FUNCTION OF BROWNIAN PARTICLES IN A MASSIVE VECTOR FIELD

Let t and x^1, x^2 be respectively the time and a set of coordinates on the two dimensional plane \mathbf{R}^2 . A point in \mathbf{R}^2 is denoted with the related radius vector $\mathbf{r} = (x^1, x^2)$. The starting point of the discussion is the following differential equation:

$$\left[\frac{\partial}{\partial t} - g(\nabla - b\mathbf{A})^2 \right] \Psi(t; \mathbf{r}, \mathbf{r}_0, \mathbf{A}) = \delta(t)\delta(\mathbf{r} - \mathbf{r}_0) \quad (1)$$

where $\mathbf{A}(\mathbf{r})$ is a time independent vector field, whose action will be specified later. For the moment, we note that the above equation is invariant under the local transformations:

$$\Psi(t; \mathbf{r}, \mathbf{r}_0, \mathbf{A}) = e^{-b(\gamma(\mathbf{r}) - \gamma(\mathbf{r}_0))} \Psi'(t; \mathbf{r}, \mathbf{r}_0, \mathbf{A}') \quad \mathbf{A}(\mathbf{r}) = \mathbf{A}'(\mathbf{r}) + \nabla\gamma(\mathbf{r}) \quad (2)$$

in which $\gamma(\mathbf{r})$ is an arbitrary function of \mathbf{r} . The solution of Eq. (1) may be expressed in terms of a Feynman path integral:

$$\Psi(t; \mathbf{r}, \mathbf{r}_0, \mathbf{A}) = \theta(t) \int_{\substack{\mathbf{R}(t)=\mathbf{r} \\ \mathbf{R}(0)=\mathbf{r}_0}} \mathcal{D}\mathbf{R}(\sigma) \exp \left\{ - \int_0^t d\sigma \left[\frac{1}{4g} \dot{\mathbf{R}}^2 + b\dot{\mathbf{R}} \cdot \mathbf{A} \right] \right\} \quad (3)$$

$\theta(t)$ being the Heaviside function: $\theta(t) = 0$ if $t < 0$ and $\theta(t) = 1$ if $t \geq 0$. Eq. (3) is related to the canonical partition function of a particle interacting with a vector potential \mathbf{A} and performing a random walk in the plane. Following Ref. [25], we introduce now the generating functional of the correlation functions of the field Ψ :

$$\Xi[J] = \left\langle \exp \left[\int dt d\mathbf{r} J(t, \mathbf{r}) \Psi(t; \mathbf{r}, \mathbf{r}_0, \mathbf{A}) \right] \right\rangle_{\mathbf{A}} \quad (4)$$

In Eq. (4) the average over the vector potential \mathbf{A} is taken according to the following prescription:

$$\langle \dots \rangle_{\mathbf{A}} = \int \mathcal{D}\mathbf{A} e^{\int d\mathbf{r} \left[\alpha(\nabla \times \mathbf{A})^2 + \frac{\mathbf{A}^2}{2} \right]} \dots \quad (5)$$

Putting

$$J(t, \mathbf{r}) = \mu \delta(t - T) \quad (6)$$

we get from Eq. (4):

$$\Xi[J]|_{J=\mu\delta(t-T)} \equiv \Xi[\mu] = \left\langle \exp \left[\mu \int d\mathbf{r} \Psi(T; \mathbf{r}, \mathbf{r}_0, \mathbf{A}) \right] \right\rangle_{\mathbf{A}} \quad (7)$$

Expanding the exponential in the right hand side of Eq. (7) we have:

$$\exp \left[\mu \int d\mathbf{r} \Psi(T; \mathbf{r}, \mathbf{r}_0, \mathbf{A}) \right] = \sum_{N=0}^{\infty} \frac{1}{N!} Z_N(T; \mathbf{r}_0, \mathbf{A}) \mu^N \quad (8)$$

where

$$Z_N(T; \mathbf{r}_0, \mathbf{A}) = \left(\int d\mathbf{r} \Psi(T; \mathbf{r}, \mathbf{r}_0, \mathbf{A}) \right)^N \quad (9)$$

The quantity $Z = \int d\mathbf{r} \Psi(T; \mathbf{r}, \mathbf{r}_0, \mathbf{A})$ has the meaning of the canonical partition function of a particle diffusing from a fixed point \mathbf{r}_0 to any other point in the plane during the time T .

From the above discussion it turns out that $\Xi[\mu]$ can be interpreted as the grand canonical partition function of a system of indistinguishable particles which perform a Brownian walk while they are interacting with a massive vector field \mathbf{A} . The parameter μ plays the role of the chemical potential.

III. THE MASSIVE VECTOR FIELD THEORY

Hereafter we will conform our notation to that used in the case of Euclidean field theories in natural units $\hbar = c = 1$. Points in \mathbf{R}^2 will be denoted with the symbols x, y, z, \dots , where $x = x^1, x^2, y = y^1, y^2$ etc. Moreover, the spatial components of vectors will be labeled using middle Greek indices $\mu, \nu, \dots = 1, 2$. The summation convention $u_\mu v^\mu = u_1 v^1 + u_2 v^2$ will be used. The volume measure $d\mathbf{r}$ is replaced by d^2x , the symbol $(x - x_0)^2$ denotes the scalar product $(x - x_0)_\mu (x - x_0)^\mu$ and similarly $\partial^2 = \partial_\mu \partial^\mu$ is the laplacian and so on.

The main subject of this Section is a model of three dimensional non-relativistic scalar fields $\psi_1(t, x), \psi_2(t, x)$ coupled with a two dimensional massive vector field $A_\mu(x)$. The dimensions in natural units of these fields are respectively:

$$[\psi_1] = 0 \quad [\psi_2] = -3 \quad [A_\mu] = -1 \quad (10)$$

The partition function of the model is given by:

$$\mathcal{Z}_{\alpha,T}[J_1, J_2] = \int \mathcal{D}A \mathcal{D}\psi_1 \mathcal{D}\psi_2 \exp [-S_\alpha(A, \psi_1, \psi_2)] \quad (11)$$

where the action $S_\alpha(A, \psi_1, \psi_2)$ is:

$$\begin{aligned} S_{\alpha,T}(A, \psi_1, \psi_2) &= \int d^2x \left[\alpha \frac{F_{\mu\nu}^2}{4} + \frac{A_\mu A^\mu}{2} \right] + \int dt d^2x [-iJ_1\psi_1 + J_2\psi_2] \\ &\quad + iT \int dt d^2x \left[\psi_1 \frac{\partial \psi_2}{\partial t} + g(\partial_\mu + bA_\mu)\psi_1(\partial^\mu - bA^\mu)\psi_2 \right] \end{aligned} \quad (12)$$

and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. In the above equation J_1 and J_2 represent external currents, while α and T are real and positive parameters. Both T and the coupling constant g have the dimension of a length, while α has the dimension of a squared mass. The connection with the grand canonical partition function of particles subjected to a Brownian motion is in the right hand side of Eq. (12). For suitable choices of the currents J_1 and J_2 the integration over the fields ψ_1 and ψ_2 reproduces exactly the grand canonical partition function of Eq. (7).

Clearly, the free action of the massive vector fields $S_{mvf}^0(A) = \int d^2x \left[\alpha \frac{F_{\mu\nu}^2}{4} + \frac{A_\mu A^\mu}{2} \right]$ corresponds to the action of the vector field $\mathbf{A}(\mathbf{r})$ in Eq. (5). Moreover, in the action (12) the field ψ_1 plays the role of a Lagrange multiplier which imposes the constraint:

$$T \left[\frac{\partial}{\partial t} - g(\partial - bA)^2 \right] \psi_2 = J_1 \quad (13)$$

The above equation becomes equal to the differential equation (1) upon making for the current J_1 the special choice

$$J_1(t, x) = \frac{T}{g} \delta(t) \delta(x - x_0) \quad (14)$$

and rescaling the field ψ_2 by the constant g , so that $\Psi = g\psi_2$. As a consequence the partition function $Z_{\alpha,T}[J_1, J_2]$ defined by Eqs. (11) and (12) is equivalent to the generating functional of Eq. (4).

We are interested to study the limit in which both α and T approach infinity. It is easy to check that the limit $\alpha \rightarrow +\infty$ imposes in the partition function (11) the constraint:

$$\epsilon^{\mu\nu} \partial_\mu A_\nu = 0 \quad (15)$$

where $\epsilon^{\mu\nu}$ is the totally antisymmetric tensor defined according to the usual convention $\epsilon^{12} = 1$. As a matter of fact, remembering that in two dimensions $F_{\mu\nu}^2 = 2F_{12}^2$, it is possible to apply to Eq. (11) the following Gaussian identity:

$$\exp \left[-\alpha \int d^2x \frac{F_{\mu\nu}^2}{4} \right] = \int \mathcal{D}\lambda \exp \left[- \int d^2x \left(i\lambda F_{12} + \frac{\lambda^2}{\alpha} \right) \right] \quad (16)$$

If α goes to infinity, in the right hand side of Eq. (16) the field $\lambda = \lambda(x)$ becomes a Lagrangian multiplier which imposes exactly condition (15), because $F_{12} = \epsilon^{\mu\nu}\partial_\mu A_\nu$. Thus, when α goes to infinity only the longitudinal component of the field \mathbf{A} survives due to the identity (15), which eliminates the transverse components. The meaning of the limit $T \rightarrow +\infty$ will become clear later, as we will introduce a second Gaussian identity resulting after the integration of the fields ψ_1 and ψ_2 .

IV. THE LIMIT $\alpha = +\infty$

In this Section we consider the model described in the previous Section in the limit $\alpha = +\infty$. As it has been discussed earlier, in this case the condition (15) forces the field A_μ to have just the longitudinal component, i. e. $A_\mu = \partial_\mu\varphi$. As a consequence, the partition function (11) and the action (12) become respectively:

$$\mathcal{Z}_{\infty,T}[J_1, J_2] = \mathcal{C} \int \mathcal{D}\varphi \mathcal{D}\psi_1 \mathcal{D}\psi_2 \exp [-S_{\infty,T}(\varphi, \psi_1, \psi_2)] \quad (17)$$

and

$$\begin{aligned} S_{\infty,T}(\varphi, \psi_1, \psi_2) &= \int d^2x \left[\frac{\partial_\mu\varphi\partial^\mu\varphi}{2} \right] + \int dt d^2x [-iJ_1\psi_1 + J_2\psi_2] \\ &\quad + iT \int dt d^2x \left[\psi_1 \frac{\partial\psi_2}{\partial t} + g(\partial_\mu + b\partial_\mu\varphi)\psi_1(\partial^\mu - b\partial^\mu\varphi)\psi_2 \right] \end{aligned} \quad (18)$$

In Eq. (17) $\mathcal{C} = \det \partial_\mu$ is an irrelevant constant which appears because of the change of measure $\mathcal{D}(\partial_\mu\varphi) \rightarrow \mathcal{D}\varphi$ and will be omitted in the following.

At this point we perform in the partition function (17) the field redefinitions: $\psi_1 = e^{-b\varphi}\psi'_1$ and $\psi_2 = e^{b\varphi}\psi'_2$. As a result we obtain:

$$\mathcal{Z}_{\infty,T}[J_1, J_2] = \int \mathcal{D}\varphi \mathcal{D}\psi'_1 \mathcal{D}\psi'_2 \exp [-S_{\infty,T}(\varphi, \psi'_1, \psi'_2)] \quad (19)$$

with

$$\begin{aligned} S_{\infty,T}(\varphi, \psi'_1, \psi'_2) &= \int d^2x \left[\frac{\partial_\mu\varphi\partial^\mu\varphi}{2} \right] + \int dt d^2x \left[-iJ_1 e^{-b\varphi}\psi'_1 + J_2 e^{b\varphi}\psi'_2 \right] \\ &\quad + iT \int dt d^2x \left[\psi'_1 \frac{\partial\psi'_2}{\partial t} + g(\partial_\mu + b\partial_\mu\varphi)\psi'_1(\partial^\mu - b\partial^\mu\varphi)\psi'_2 \right] \end{aligned} \quad (20)$$

The action (20) is Gaussian in the fields ψ'_1 and ψ'_2 and it is thus possible to eliminate these fields with the help of a simple Gaussian integration in the partition function (19). To this

purpose, we have to compute the path integral:

$$\begin{aligned} \mathcal{Z}_T[\varphi, J_1, J_2] = & \\ \int \mathcal{D}\psi'_1 \mathcal{D}\psi'_2 \exp \left\{ - \int dt d^2x \left[iT \left(\psi'_1 \frac{\partial \psi'_2}{\partial t} + g \partial_\mu \psi'_1 \partial^\mu \psi'_2 \right) - ie^{-b\varphi} J_1 \psi'_1 + e^{b\varphi} J_2 \psi'_2 \right] \right\} \end{aligned} \quad (21)$$

Here with the symbol $\mathcal{Z}_T[\varphi, J_1, J_2]$ we have denoted the part of the partition function $\mathcal{Z}_{\infty,T}[J_1, J_2]$ which contains only the fields ψ'_1 and ψ'_2 , i. e.:

$$\mathcal{Z}_{\infty,T}[J_1, J_2] = \int \mathcal{D}\varphi e^{- \int d^2x \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi} \mathcal{Z}_T[\varphi, J_1, J_2] \quad (22)$$

In the path integral (22) the field ψ'_1 plays the role of a Lagrange multiplier. After integrating it out, the generating functional $\mathcal{Z}_T[\varphi, J_1, J_2]$ becomes:

$$\mathcal{Z}_T[\varphi, J_1, J_2] = \int \mathcal{D}\psi'_2 \delta \left(T \frac{\partial \psi'_2}{\partial t} - g \Delta \psi'_2 - e^{-b\varphi} J_1 \right) \exp \left\{ - \int dt d^2x e^{b\varphi} J_2 \psi'_2 \right\} \quad (23)$$

The δ -function appearing in the above equation forces the field ψ'_2 to satisfy the equation:

$$T \frac{\partial \psi'_2}{\partial t} - g \Delta \psi'_2 = e^{-b\varphi} J_1 \quad (24)$$

At this point we make for the current J_1 the special choice:

$$J_1(t, x) = \frac{T}{g} \delta(t) \delta(x - x_0) \quad (25)$$

The solution of Eq. (24) corresponding to this choice is:

$$\psi'_2(t, x) = \theta(t) \frac{T}{4\pi g^2 t} e^{-\frac{(x-x_0)^2}{4gt}} e^{-b\varphi(x_0)} \quad (26)$$

Thus, when integrating over the field ψ'_2 in Eq.(23), we arrive at the following expression of the generating functional $\mathcal{Z}_T[\varphi, J_1, J_2]$:

$$\mathcal{Z}_T[\varphi, J_1, J_2] = \exp \left[- \int dt d^2x J_2(t, x) \frac{\theta(t) T}{4\pi g^2 t} e^{b(\varphi(x) - \varphi(x_0))} e^{-\frac{(x-x_0)^2}{4gt}} \right] \quad (27)$$

It is still possible to use the freedom to choose the second current J_2 :

$$J_2(t, x) = T \delta(t - T) \quad (28)$$

In this way we obtain the final form of the functional $\mathcal{Z}_T[\varphi, J_1, J_2]$:

$$\mathcal{Z}_T[\varphi, J_1, J_2] = \theta(T) \exp \left[- \int d^2x \frac{e^{b(\varphi(x) - \varphi(x_0))}}{4\pi g^2} e^{-\frac{(x-x_0)^2}{4gT}} \right] \quad (29)$$

More generally, what we have proved here is the following identity:

$$\begin{aligned} & \int \mathcal{D}\psi_1 \mathcal{D}\psi_2 \exp \left\{ - \int dt d^2x \left[iT \left(\psi_1 \frac{\partial \psi_1}{\partial t} + g(\partial_\mu + b\partial_\mu \varphi)(\partial^\mu - b\partial^\mu \varphi)\psi_2 \right) + J_2 \psi_2 - iJ_1 \psi_1 \right] \right\} \\ &= \exp \left\{ - \int dt d^2x \int dt' d^2x' \left[e^{b(\varphi(x)-\varphi(x'))} \frac{\theta(t-t')}{4\pi g T(t-t')} e^{\frac{(x-x')^2}{4g(t-t')}} J_1(t', x') J_2(t, x) \right] \right\} \end{aligned} \quad (30)$$

of which Eq. (29) is a particular case. Putting the result of Eq. (29) back in Eq. (22), we find:

$$\mathcal{Z}_{\infty, T}[J_1, J_2] = \int \mathcal{D}\varphi \exp \left\{ - \int d^2x \left[\frac{(\partial_\mu \varphi)^2}{2} + e^{b(\varphi(x)-\varphi(x_0))} \frac{\theta(T)}{4\pi g^2} \exp \left(-\frac{(x-x_0)^2}{4gT} \right) \right] \right\} \quad (31)$$

V. THE LIMIT $T \rightarrow \infty$ AND LIOUVILLE FIELD THEORY

In this Section the limit $T \rightarrow +\infty$ is taken in the partition function (31). In this limit it is possible to put $\theta(T) = 1$ since $T > 0$. Using the fact that $\lim_{T \rightarrow +\infty} \exp \left\{ -\frac{(x-x_0)^2}{4gT} \right\} = 1$ we arrive in this way at the partition function:

$$\mathcal{Z}_{\infty, \infty}[J_1, J_2] = \int \mathcal{D}\varphi \exp \left\{ - \int d^2x \left[\frac{(\partial_\mu \varphi)^2}{2} + \frac{1}{4\pi g^2} e^{b(\varphi(x)-\varphi(x_0))} \right] \right\} \quad (32)$$

Finally, performing the shift of fields $\varphi'(x) = \varphi(x) - \varphi(x_0)$, one finds the partition function of the Liouville model with the additional condition

$$\varphi'(x_0) = 0 \quad (33)$$

It is easy to check that in the limits $b = 0$ and $g \rightarrow +\infty$ the partition function in Eq. (11) becomes trivial, as it is expected from the equivalence with the above Liouville field theory.

VI. CONCLUSIONS

In this work the Liouville field theory has been mapped into the two component scalar field theory interacting with a massive vector field defined by Eqs. (11) and (12). The two theories have been proved to be equivalent. The massive vector field model after the limit $\alpha \rightarrow +\infty$ may be considered in some sense as a sort of BF model [26] where just the longitudinal modes propagate, since the gauge invariance is broken by the mass-term [36]. The most important ingredient in our procedure is the Gaussian formula (30). This formula allows to express the exponential interaction term in the Liouville action in the form of the

generating functional of two scalar fields in three dimensions. Eq. (30) has been obtained after the field rescaling $\psi_1 = e^{-b\varphi}\psi'_1$ and $\psi_2 = e^{b\varphi}\psi'_2$. This procedure may in principle alter the functional integral measure, possibly spoiling our result. For this reason, in the Appendix Eq. (30) has been re-derived using an alternative method, which does not involve the rescaling of the fields.

The model of Eqs. (11) and (12) has polynomial interactions and can be treated by standard field theoretical techniques. If one uses for instance the method of Ref. [27], the massive vector field \mathbf{A} can be eliminated, leaving as a result a non-local and multi-component scalar field theory. Despite the non-locality, theories of this kind may be investigated with the help of approximations like RPA or Hartree–Fock, or with the techniques of strong coupling [28]. One limitation of our procedure is that the Liouville field φ is mapped into the longitudinal component of the vector field \mathbf{A} . For this reason, the potential obtained in Eq. (32) depends on the difference of fields $\varphi(x) - \varphi(x_0)$. For the same reason, it is only possible to compute the correlation functions of differences of Liouville fields. Alternatively, one may require that the field φ satisfies the condition (33) at an arbitrarily chosen point x_0 as it has been done in this work.

Let us note that the derivation of Eq. (30), as well as the whole procedure used in order to obtain the theory of Liouville from the model of Eqs. (11) is not just a sequence of formal passages. It is actually based on the way in which in statistical mechanics one passes from the canonical partition function to the grand canonical partition function, as it was briefly explained in Section II. Additionally, the grand canonical partition function has been written with the help of scalar fields, following path integral techniques used in many-body physics [29]. For this reason, our approach is not limited to the Euclidean two dimensional space, but it may be extended to any n -dimensional manifold on which the solution of Eq. (1), expressing the Green function of a Brownian particle immersed in an external magnetic field, is known. Of course, the Liouville action will be no longer renormalizable if $n > 2$. This is consistent with the fact that, within our procedure, the theory of Liouville is mapped to a massive vector field theory, which is also nonrenormalizable in dimensions higher than two. In extending our approach to general manifolds, one should be aware that the Gaussian formula (30) is modified by the geometry and/or by the presence of zero modes. Already in the case of an n -dimensional Euclidean space, the Green function $\Psi(t; \mathbf{r}, \mathbf{r}_0, \mathbf{A})$ satisfying Eq. (1) has a behavior with respect to the time t which depends on n . Therefore, if we wish

to obtain an analogue of the Liouville partition function in n -dimensions after performing the large time limit $T \rightarrow +\infty$ as it has been done in Section V, we need to change the constant coefficients appearing in the currents J_1, J_2 . On a general manifold the situation is much more complicated. It is easy to adjust the left hand side of Eq. (30), which is the partition function of a field theory, to include a background metric. The same is not true however for the right hand side, which is the solution of a differential equation whose closed form is known just in the case of a few non-flat geometries. An additional difficulty can arise on spatial manifolds which admit non-trivial classical solutions of Eq. (15) consisting of harmonic zero modes. On compact Riemann surfaces the explicit form of these non-trivial classical solutions may be found in Refs. [30, 31]. The problem is that the limit $\alpha \rightarrow +\infty$ does not project out the harmonic zero modes from the action (18), so that one should eliminate them manually. In gauge field theories the harmonic zero modes may be regarded as gauge degrees of freedom and gauged away using a method based on BRST techniques proposed by Polyakov [32] and further developed by the authors of Refs. [33, 34]. In the present context, however, this strategy cannot be applied. As a matter of fact, the gauge symmetry (2) is explicitly broken by the mass term of the gauge fields and by the insertion of the external currents J_1, J_2 , which are needed in order to obtain the theory of Liouville in its final form. If we attempt to gauge away the harmonic zero modes with a transformation similar to the field rescaling which has been used in order to obtain Eq. (19), they will remain in the action due to the current terms.

In conclusion, the extension of the present approach to nontrivial manifolds is a difficult task, in particular because the explicit expressions of the canonical and of the grand canonical partition functions are not known in the case of a Brownian particle immersed both in a magnetic field and in a gravitational field with an arbitrary metric. As a consequence, it is not possible to write down an analogue of the fundamental identity (30) on general manifolds. What it is however feasible, is the generalization of the Gaussian formula (30) to include other theories than the Liouville model. Work is in progress in that direction.

APPENDIX A: AN ALTERNATIVE PROOF OF EQ. (31)

In this Appendix we provide an alternative method to perform the integration over the fields ψ_1, ψ_2 in the partition function $Z_{\infty, T}[J_1, J_2]$ of Eq. (17). To this purpose, we isolate

from the expression of $Z_{\infty,T}[J_1, J_2]$ only the part in which the fields ψ_1, ψ_2 are involved:

$$\begin{aligned} Z_\psi = & \int \mathcal{D}\psi_1 \mathcal{D}\psi_2 \exp \left\{ -iT \int dt d^2x \left[\psi_1 \frac{\partial \psi_2}{\partial t} + g D_{+\mu} \psi_1 D_-^\mu \psi_2 \right] \right\} \\ & \times \exp \int dt d^2x [iJ_1 \psi_1 - J_2 \psi_2] \end{aligned} \quad (\text{A1})$$

with

$$D_{+\mu} = \partial_\mu + b \partial_\mu \varphi \quad D_-^\mu = \partial^\mu - b \partial^\mu \varphi \quad (\text{A2})$$

Since the action in (A1) is at most quadratic in the fields ψ_1, ψ_2 , it is convenient to perform the shift of variables:

$$\psi_1(t, x) = \psi'_1(t, x) + i \int dt'' d^2x'' G(t'' - t; x'' - x; \varphi) J_2(t'', x'') \quad (\text{A3})$$

$$\psi_2(t, x) = \psi'_2(t, x) + \int dt' d^2x' G(t - t'; x - x'; \varphi) J_1(t', x') \quad (\text{A4})$$

where

$$G(t - t'; x - x'; \varphi) = \langle \psi_1(t', x') \psi_2(t, x) \rangle \quad (\text{A5})$$

is the two point function of the fields ψ_1, ψ_2 :

$$G(t - t'; x - x'; \varphi) = e^{b(\varphi(x) - \varphi(x'))} \frac{\theta(t - t')}{4\pi g T(t - t')} e^{-\frac{(x - x')^2}{4g(t - t')}} \quad (\text{A6})$$

After the substitution (A3–A4), the partition function Z_ψ becomes:

$$Z_\psi = \det K e^{- \int d^2x \frac{\theta(T)}{4\pi g^2} e^{-\frac{(x-x_0)^2}{4gT}} e^{b(\varphi(x) - \varphi(x_0))}} \quad (\text{A7})$$

with

$$\det K = \det \left[-iT \left(\frac{\partial}{\partial t} - g D_{-\mu} D_-^\mu \right) \right]^{-1} \quad (\text{A8})$$

In order to obtain Eq. (A7) we have used the special values of the currents J_1, J_2 given in Eqs. (25) and (28) respectively. Substituting the above result for Z_ψ in the partition function $Z_{\infty,T}[J_1, J_2]$ we find:

$$\begin{aligned} Z_{\infty,T}[J_1, J_2] = & \int \mathcal{D}\varphi e^{- \int d^2x \frac{(\partial_\mu \varphi)^2}{2}} \det K \\ & \times \exp \left\{ - \int d^2x \frac{\theta(T)}{4\pi g^2} e^{-\frac{(x-x_0)^2}{4gT}} e^{b(\varphi(x) - \varphi(x_0))} \right\} \end{aligned} \quad (\text{A9})$$

Apart from the presence of the determinant of the differential operator K , the expression of the partition function $Z_{\infty,T}[J_1, J_2]$ is equal to that reported in Eq. (31). In principle, this

determinant is dependent on the field φ and may thus lead to a theory which is not that of Liouville. To show that this is not true because $\det K$ is trivial, we follow an analogous proof provided in the case of non-relativistic scalar electrodynamics in Ref. [24]. First of all, we write the determinant as a path integral in the fields ψ'_1, ψ'_2 :

$$\begin{aligned} \det K = & \int \mathcal{D}\psi'_1 \mathcal{D}\psi'_2 \exp \left\{ iT \int dt d^2x \left[\psi'_1 \left(\frac{\partial}{\partial t} - g\partial_\mu^2 \right) \psi'_2 \right] \right\} \\ & \times \exp \left\{ iT \int dt d^2x \left[gb\partial_\mu\varphi(\psi'_1 \partial^\mu \psi'_2 + \partial_\mu \psi'_1 \psi'_2) - gb^2(\partial_\mu\varphi)^2 \psi'_1 \psi'_2 \right] \right\} \quad (\text{A10}) \end{aligned}$$

Here we have separated the contributions appearing in the action according to the different powers of the coupling constant b . In evaluating $\det K$, the field φ may be regarded as an external field. The action appearing in the exponent of Eq. (A10) produces the Feynman rules shown in Fig. 1. It is easy to realize that the Feynman diagrams which contribute to

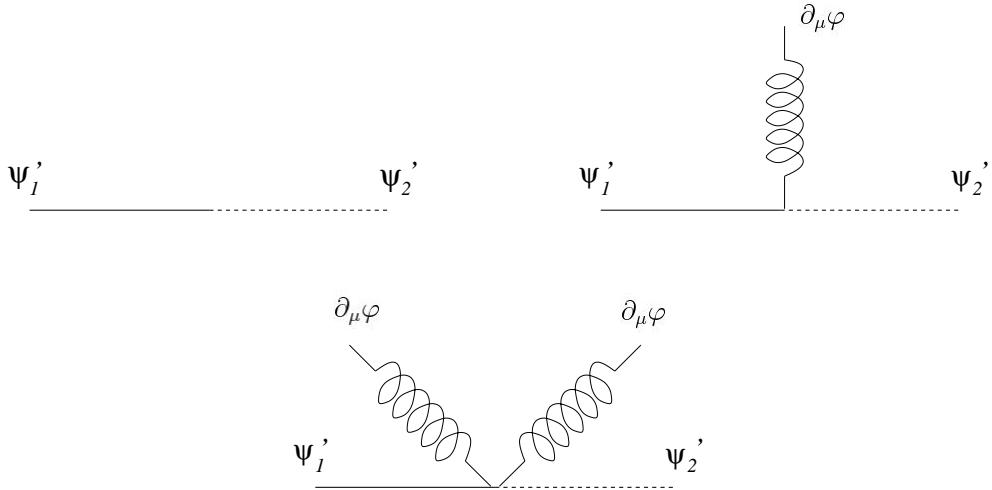


FIG. 1: Feynman diagrams of the ψ'_1, ψ'_2 action appearing in Eq. (A10). Full lines and dashed lines propagate the fields ψ'_1 and ψ'_2 respectively, while wavy lines correspond to the insertion of the external field φ .

$\det K$ are loops with n external legs. The external legs correspond to the insertion of the external fields φ , see Fig. 2. Moreover, the loops consist of dashed internal lines alternated by full internal lines. This is due to the fact that the free propagator

$$\langle \psi'_1(t', x') \psi'_2(t, x) \rangle = \frac{\theta(t - t')}{4\pi g T(t - t')} e^{-\frac{(x-x')^2}{4g(t-t')}} \quad (\text{A11})$$

allows only the contractions of ψ'_1 fields with ψ'_2 fields. Most important, the propagator (A11) is multiplied by the Heaviside function $\theta(t - t')$. It is thus easy to convince oneself

that the loops which contribute to $\det K$ vanish identically because, after the contractions of the fields ψ'_1 with the fields ψ'_2 are performed, there is no common interval of time in which all the resulting θ -functions could be simultaneously different from zero. As a consequence, $\det K = 1$ and the expression of the partition function $Z_{\infty,T}[J_1, J_2]$ given in Eq. (A9) coincides with that of Eq. (31), ending our proof.

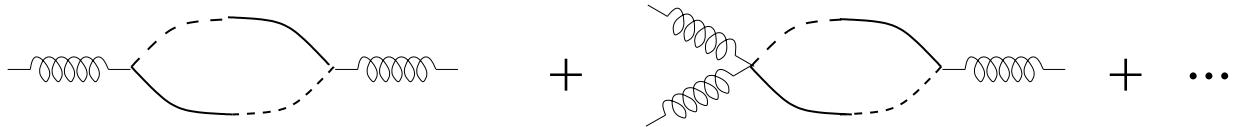


FIG. 2: Loop expansion of $\det K$.

Acknowledgments

This work has been financed in part with funds allocated for the years 2006-2008 by the Polish Ministry of Science and University (Ministerstwo Nauki i Szkolnictwa Wyższego) in the frame of the scientific project "Applications of supersymmetric and topological field theories to the description of realistic polymer systems", Scientific Grant no. N202 156 31/2933. The authors would also like to thank Tomasz Wydro and the LPMC, Université Paul Verlaine-Metz, where the last stage of this work was performed, for the nice hospitality.

Last but not least F. Ferrari is indebted to C. Mudry and to M. Matone for pointing out the interesting references [5, 6, 7].

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- [35] We would like to mention at this point the fact that a perturbative approach may be defined also for the quantum theory of Liouville, see [22, 23].
- [36] We thank the anonymous referee of *Phys. Lett. B* for this important comment and for other helpful hints.